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A Quantitave Version of the Schneider-Lang Theorem

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§1. Introduction

We begin with the result of Hermite and Lindemann which implies the transcendence of e and π. Secondly, we refer to the Gel'fond-Schneider theorem, that is the solution of the seventh problem of Hilbert, settled in 1934. Next, we state the Schneider-Lang theorem, and we show how this result contains the Hermite-Lindemann theorem and the Gel'fond-Schneider theorem. Finally, here, we give a quantitative version of the theorem of Schneider and Lang.

Let us recall the above theorems.

Theorem (Hermite-Lindemann). *Let α be a non-zero algebraic number, then e^{α} is a transcendental number.*

Theorem (Gel'fond-Schneider). *Let α and β be algebraic numbers. Assume that $\alpha \neq 0$, $\alpha \neq 1$, and that β is not rational.*

Let $\log \alpha$ *be any determination of the logarithm of* α. *Then* $\alpha^{\beta} = \exp(\beta \log \alpha)$ *is a transcendental number.*

Important examples of numbers whose transcendence follows from these theorems are e^{π} and $2^{\sqrt{2}}$. Gel'fond's proof was based on the following ideas. If we suppose $\log \alpha_2 = \beta \log \alpha_1$ with an irrational algebraic number β and algebraic numbers α_1, α_2 ($\neq 0, 1$), then the two functions $\alpha_1^z = e^{z \log \alpha_1}$ and $\alpha_2^z = e^{z \log \alpha_2} = e^{z\beta \log \alpha_1}$ take algebraic values at all integer points. Moreover, their derivatives with respect to the derivation operator $\frac{1}{\log \alpha_1} \frac{d}{dz}$ at all integer points are also algebraic.
We can construct a non-zero polynomial $P \in \mathbf{Z}[X_1, X_2]$ such that the function $F(z) = P(\alpha_1^z, \alpha_2^z)$ vanishes at several integer points with a high order, and after that we show that F has more and more zeroes, finally, $F \equiv 0$, which is a contadiction to the algebraic independence of the two functions α_1^z and α_2^z because β is not rational.

We now mention the following general result.

Theorem (Schneider-Lang). *Let* K *be a number field and let* $f_1, \cdots, f_h$ *be meromorphic functions. We assume that* f_1 *and* f_2 *are algebraically independent over* $\mathbf{Q}$, *and of order at most*

ρ_1, ρ_2 *respectively. We assume further that the ring* $K[f_1,\cdots,f_h]$ *is stable under the derivation* $\frac{d}{dz}$. *Then the set of complex numbers* w, *which are not poles of* $f_1,\cdots,f_h$ *and such that* $f_j(w) \in K$ *for* $1 \leqq j \leqq h$ *is finite with at most* $(\rho_1 + \rho_2)[K:\mathbf{Q}]$ *elements.*

The *order* of an entire function f is

$$\limsup_{R\to\infty} \frac{\log \log |f|_R}{\log R}$$

and if a meromorphic function can be written as quotient of two entire functions of order $\leqq \rho$, then it is called a function of *order* $\leqq \rho$.

We obtain the theorem of Hermite-Lindemann as a corollary to this theorem by setting $K = \mathbf{Q}(\alpha, e^{\alpha})$, $h = 2$, $f_1(z) = z$, $f_2(z) = e^z$, $\rho_1 = 0$, $\rho_2 = 1$. Secondly, when $K = \mathbf{Q}(\alpha_1, \alpha_2, \beta)$, $h = 2$, $f_1(z) = e^z$, $f_2(z) = e^{\beta z}$, $\rho_1 = \rho_2 = 1$, we deduce Gel'fond-Schneider theorem from this theorem immediately.

This upper bound $(\rho_1 + \rho_2)[K:\mathbf{Q}]$ is sometimes the best possible (the functions z and $\exp(z(z-1)\cdots(z-k+1))$ take integer values at k points).

The Gel'fond-Schneider theorem shows that if we take any distinct complex numbers $w_1,\cdots,w_m$ with $m > (\rho_1 + \rho_2)[K:\mathbf{Q}]$ then at least one value $f_i(w_\mu)$ $(1 \leqq i \leqq h,\ 1 \leqq \mu \leqq m)$ doesn't belong to K. We deduce from this property that the sum

$$\sum_{i=1}^{h} \sum_{\mu=1}^{m} |f_i(w_\mu) - \alpha_{i\mu}|$$

is not zero where $\alpha_{i\mu}$ are any algebraic numbers in K. Here, we give a lower estimate to this sum in terms of heights $H(\alpha_{i\mu})$ of $\alpha_{i\mu}$.

§2. Statement of result

Theorem. *Let* K *be a number field and let* $f_1, \cdots, f_h$ *be meromorphic functions. We assume that* f_1 *and* f_2 *are algebraically independent over* $\mathbf{Q}$, *and of order at most* ρ_1, ρ_2 *respectively. We assume further that the ring* $K[f_1, \cdots, f_h]$ *is stable under the derivation* $\frac{d}{dz}$. *We take any distinct complex numbers* $w_1, \cdots, w_m$ *which are not poles of* $f_1, \cdots, f_h$ *with* $m > (\rho_1 + \rho_2)[k:\mathbf{Q}]$ *and* $w_1 = 0$. *Suppose also that* $f_i(0) \in K$ $(1 \leqq i \leqq h)$. *We denote by* d *the maximum of the total degrees of* A_i *where* $\frac{d}{dz} f_i = A_i(f_1, \cdots, f_h)$, $A_i \in K[X_1, \cdots, X_h]$ *for* $1 \leqq i \leqq h$. *Put* $\delta = [K:\mathbf{Q}]$ *and*

$$\kappa_0 = \frac{2^h(d-1)(\delta-1)(\rho_1+\rho_2)}{m-\delta(\rho_1+\rho_2)}.$$

Then for all $\kappa > \kappa_0$, *there exists an explicit number* H_0 *such that if we take any algebraic numbers* $\alpha_{i\mu}$ $(1 \leqq i \leqq h,\ 1 \leqq \mu \leqq m)$ *in* K, *then we have*

$$\sum_{i=1}^{h} \sum_{\mu=1}^{m} |f_i(w_\mu) - \alpha_{i\mu}| \geqq \exp(-H^\kappa)$$

where $H = \max\,(H(\alpha_{i\mu}), H_0)$.

Remark. We deduce from this theorem the following result which is mentioned by D.W.Masser in [M]: For any $\varepsilon > 0$ there exists a number $m_0(\varepsilon)$ such that for all $m > m_0(\varepsilon)$ we have

$$\sum_{i=1}^{h} \sum_{\mu=1}^{m} |f_i(w_\mu) - \alpha_{i\mu}| \geq \exp(-H^{\varepsilon}).$$

The above theorem gives to Masser's $m_0(\varepsilon)$ an explicit value, namely

$$m_0(\varepsilon) = \frac{2^h(d-1)(\delta-1)(\rho_1+\rho_2)}{\varepsilon} + \delta(\rho_1 + \rho_2).$$

§3. Outline of the proof

We assume

$$(1) \qquad \sum_{i=1}^{h} \sum_{\mu=1}^{m} |f_i(w_\mu) - \alpha_{i\mu}| < \exp(-H^{\kappa})$$

and we shall get a contradiction. Let ℓ be a sufficiently large integer. Without loss of generality we may assume $H \gg H_0$ where H_0 is sufficiently large with respect to ℓ.
Put

$$U = H^{\kappa}, \quad T = H^{\kappa/2^h}, \quad L_1 = \ell T^{\rho_2/(\rho_1+\rho_2)}, \quad L_2 = \ell T^{\rho_1/(\rho_1+\rho_2)}.$$

Step 1. *Construction of algebraic numbers* $\alpha(\lambda_1,\lambda_2,t,\mu)$

We can write

$$\frac{d^t}{dz^t} f_1^{\lambda_1} f_2^{\lambda_2} = Q_{\lambda_1,\lambda_2,t}(f_1,\cdots,f_h)$$

where $Q_{\lambda_1,\lambda_2,t}$ is a polynomial with coefficients in K ($0 \leqq \lambda_1 < L_1$, $0 \leqq \lambda_2 < L_2$, $0 \leqq t$), using the differential equations of f_i ($1 \leqq i \leqq h$). Define

$$\alpha(\lambda_1,\lambda_2,t,\mu) = Q_{\lambda_1,\lambda_2,t}(\alpha_{1\mu},\cdots,\alpha_{h\mu}).$$

Then for $0 \leqq t < T$ we get

$$\left|\frac{d^t}{dz^t} f_1^{\lambda_1} f_2^{\lambda_2}(w_\mu) - \alpha(\lambda_1,\lambda_2,t,\mu)\right| < e^{-\frac{1}{2}U}$$

by the hypothesis (1).

Step 2. *Construction of an auxiliary function* F

We consider the linear system

$$\sum_{\lambda_1}\sum_{\lambda_2} p_{\lambda_1\lambda_2}\,\alpha(\lambda_1,\lambda_2,t,\mu) = 0$$

for $0 \leqq \lambda_1 < L_1$, $0 \leqq \lambda_2 < L_2$, $0 \leqq t < T$, $1 \leqq \mu \leqq m$ of Tm equations in L_1L_2 unknowns $p_{\lambda_1\lambda_2}$. This system has coefficients in K and we get a non-trivial solution by the choice of L_1 and L_2. Siegel's lemma gives that

$$\max_{\substack{0 \leqq \lambda_1 < L_1 \\ 0 \leqq \lambda_2 < L_2}} \log |p_{\lambda_1\lambda_2}| < \frac{1}{\ell} T \log T + O(T).$$

Put

$$F = \sum_{\lambda_1}\sum_{\lambda_2} p_{\lambda_1\lambda_2} f_1^{\lambda_1} f_2^{\lambda_2}(z).$$

Step 3. *Upper bound of the order of zeroes at the origin*

Liouville's theorem implies that $f_i(0) = \alpha_{i1}$ for all $1 \leqq i \leqq h$ because $f_i(0)$ is algebraic. From this, we have

$$\frac{d^t}{dz^t} F(0) = \sum_{\lambda_1} \sum_{\lambda_2} p_{\lambda_1 \lambda_2} \alpha(\lambda_1, \lambda_2, t, 1) = 0$$

for $0 \leqq t < T$, that means $\mathrm{ord}_{z=0} F \geqq T$.

Let T_1 be the smallest integer such that there exists $1 \leqq \mu_0 \leqq m$ with

$$\sum_{\lambda_1} \sum_{\lambda_2} p_{\lambda_1 \lambda_2} \alpha(\lambda_1, \lambda_2, T_1, \mu_0) \neq 0.$$

By the algebraic independence of f_1 and f_2, we can deduce from the theorem of Brownawell and Masser ([B-M])

$$\mathrm{ord}_{z=0} F \leqq (d\ell^2 T)^{2^{h-1}},$$

so we have

$$T_1 \leqq (d\ell^2 T)^{2^{h-1}}.$$

Step 4. *Contradiction*

We can estimate the derivatives of F:

$$\log \left| \frac{d^t}{dz^t} F(w_\mu) \right| < -\frac{1}{3} U$$

for $0 \leqq t < T_1$, $1 \leqq \mu \leqq m$.

Then using the residue formula, we have

$$\log |F|_r \leqq \left(\frac{1}{\ell} - \frac{m}{\rho_1 + \rho_2}\right) T_1 \log T_1 + O(T_1)$$

for $r = 1 + \max_{1 \leqq \mu \leqq m} |w_\mu|$.

Put

$$\gamma = \sum_{\lambda_1} \sum_{\lambda_2} p_{\lambda_1 \lambda_2} \alpha(\lambda_1, \lambda_2, T_1, \mu_0)$$

which is not zero. As above we can estimate the T_1th derivative of f:

$$\log \left| \frac{d^{T_1}}{dz^{T_1}} F(w_{\mu_0}) - \gamma \right| < -\frac{1}{3}U.$$

Then we get

$$|\gamma| \leqq \left| \frac{d^{T_1}}{dz^{T_1}} F(w_{\mu_0}) \right| + \left| \frac{d^{T_1}}{dz^{T_1}} F(w_{\mu_0}) - \gamma \right|$$

$$\leqq T_1^{T_1} |F|_r + e^{-\frac{1}{3}U},$$

hence we obtain

$$\log |\gamma| \leqq (1 + \frac{1}{\ell} - \frac{m}{\rho_1 + \rho_2}) T_1 \log T_1 + O(T_1).$$

However we get by the size inequality

$$\log |\gamma| \geqq -(\delta - 1)(1 + \frac{1}{\ell} + \frac{2^h(d-1)}{\kappa}) T_1 \log T_1 + O(T_1)$$

then we arrive at the contradiction:

$$\kappa \leqq \kappa_0 = \frac{2^h(d-1)(\delta-1)(\rho_1+\rho_2)}{m - \delta(\rho_1+\rho_2)}.$$

(Q.E.D.)

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